Ref. 3. This involves a cubic equation in $\sin^2\theta_w$, with surface angle and M_∞ appearing in the coefficients.

The shock-wave angles given by Eq. (10) differ from those calculated by Kopal[5] as follows: for $M_\infty = 3$ and $\theta_c = 42°$, difference less than 1°; for $M_\infty > 3$ and $\theta_c < 42°$, difference less than $\frac{1}{2}°$; and for $\theta_c > 42°$, poor agreement.

This last restriction is not serious, since the maximum possible θ_c for attached conical shock is 49.3° at $M_\infty = 3$ and rises to 56.7° at $M_\infty = 10$.

The shock angle relation provides a possible simplification for computing methods, such as that suggested by Miles[6] for finding properties in the flow field behind a conical shock.

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Laminar Flow in Plane Wakes of a Conducting Fluid in the Presence of a Transverse Magnetic Field

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It is well known that flows in wakes arising out of the separation of the fluid from an obstacle on both sides tend to be turbulent as the Reynolds number R exceeds about 1000. This tendency increases with increasing R. In the case of a flat plate, although the boundary layer remains laminar as far as the trailing edge for $R < 10^6$, the flow in the wake still becomes turbulent. The reason for this, perhaps, lies in the circumstance that velocity profiles in the wake, all of which possess a point of inflexion, are extremely unstable. If, however, the fluid is electrically conducting and a magnetic field is present, transition to turbulence may be delayed due to the stabilizing influence of the field, and a laminar flow presumably may be maintained in the wake.

Using Oseen's approximation, Tollmien[1] obtained the velocity distribution (based on similarity considerations) in the plane laminar wake behind a drag-producing body at a large distance from the body. Recently, Mellor[2] has given a general solution to the laminar flow in wakes and has shown that Tollmien's solution is the leading term of a more general series solution and represents the asymptotic behavior of the wake at a large distance from the body. Mellor's solution includes the case of a wake behind a self-propelled body as discussed by Birkhoff and Zarantonello.[3] In this note, Tollmien's solution will be extended to the case of an electrically conducting fluid when a transverse magnetic field is present, using Mellor's approach. The induced magnetic field is neglected in the analysis, and this is justified for low magnetic Reynolds number, which often is the case in most aerodynamic applications, as pointed out by Resler and Sears.[4] Since the laminar wake is reached by fluid particles that move along streamlines passing fairly close to the body, boundary-layer approximations are valid for this stratum of fluid.

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The x axis is taken along the direction of the incident stream moving with velocity U, and a magnetic field $H_0(x)$ is applied perpendicular to this direction, with the origin somewhere in the body. As the electric field $\mathbf{E}$ satisfies curl $\mathbf{E} = 0$ in the steady state, E_z is constant (E_0, say), the z axis being at right angles to the XY plane. Inside the wake, a current is induced in the z direction due to the interaction of the fluid with the magnetic field. Neglecting the induced field and using $j_z = \sigma(E_0 - \mu u H_0)$ from the z component of Ohm's law, the body force $\mu\mathbf{j} \times \mathbf{H}$ has the following components:

$$(\mu\mathbf{j} \times \mathbf{H})_x = \sigma(E_0H_0 - \mu^2uH_0^2) \qquad (\mu\mathbf{j} \times \mathbf{H})_y = 0 \quad (1)$$

where μ and σ are the magnetic permeability and electrical conductivity respectively.

The steady boundary-layer equations are now

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} + \frac{\sigma}{\rho}(E_0H_0 - \mu^2uH_0^2) \quad (2)$$

$$0 = \partial p/\partial y \quad (3)$$

where the other components of Maxwell's equations become redundant in view of the foregoing simplifying assumptions. Determining $-(1/\rho)\cdot\partial p/\partial x$ from the freestream value and introducing Oseen's approximation $u = U + u_1$ and $v = v_1$, where u_1 and $v_1 \ll U$, Eq. (2) becomes

$$U\frac{\partial u_1}{\partial x} = \nu\frac{\partial^2 u_1}{\partial y^2} - \frac{\sigma\mu^2H_0^2(x)}{\rho}u_1 \quad (4)$$

The boundary conditions are $\partial u_1/\partial y = 0$ at $y = 0$ and $u_1 \to 0$ as $y \to \infty$.

Now seek solutions of the form

$$u_1 = a(x)\cdot f(\eta) \qquad \eta = y/C(x) \quad (5)$$

Inserting (5) in (4) yields, after some rearrangement,

$$f'' + \eta f'\cdot\left(\frac{UCC'}{\nu}\right) - \left(\frac{Ua'C^2}{\nu a}\right)f - \left(\frac{\sigma\mu^2H_0^2(x)\cdot C^2}{\rho\nu}\right)f = 0 \quad (6)$$

where a prime denotes differentiation with respect to the similarity variable η. It is clear from (6) that, for similarity solutions, coefficients within the parentheses of various terms in (6) must be constants. Hence one sets

$$UCC'/\nu = 2 \qquad UC^2a'/\nu a = -4\lambda \qquad \mu^2H_0^2(x)\cdot C^2 = C_1 \quad (7)$$

The first two equations of (7) give

$$C^2 = 4\nu x/U \qquad a = Ax^{-\lambda} \quad (8)$$

and the last equation gives

$$H_0(x) \sim x^{-1/2} \quad (9)$$

Thus a constant magnetic field evidently precludes affinely similar velocity profiles, and a similarity solution is possible only when $H_0(x)$ behaves like (9). Equation (6) now reduces to

$$f'' + 2\eta f' + 4\lambda' f = 0 \quad (10)$$

where

$$4\lambda' = 4\lambda - (\sigma c_1/\rho\nu) \quad (11)$$

Equation (10) with the boundary conditions $f' = 0$ at $\eta = 0$ and $f = 0$ at $\eta = \infty$ constitutes a Sturm-Liouiville problem with eigenfunctions $f_{\lambda'}$ and eigenvalues λ'.

Putting $\xi = \eta^2$ and $f(\eta) = g(\xi)$, Eq. (10) becomes

$$\xi g'' + (\tfrac{1}{2} + \xi)g' + \lambda' g = 0 \quad (12)$$

where the prime now denotes differentiation with respect to ξ. Equation (12) is a confluent hypergeometric equation having the following series solution:

$$g(\xi) = 1 - \frac{\lambda'}{\frac{1}{2}}\xi + \frac{\lambda'}{\frac{1}{2}} \cdot \frac{\lambda' + 1}{\frac{1}{2} + 1} \cdot \frac{\xi^2}{2!} - \frac{\lambda'}{\frac{1}{2}} \cdot \frac{\lambda' + 1}{\frac{1}{2} + 1} \cdot \frac{\lambda' + 2}{\frac{1}{2} + 2} \cdot \frac{\xi^3}{3!} + \ldots \quad (13)$$

Since $f'(\eta) = 2\eta g'(\xi)$, the boundary condition $f' = 0$ at $\eta = 0$ is satisfied by the requirement that g' be finite at the origin. The second complementary solution to Eq. (12) is not regular at the origin, and it is discarded. Equation (13) also can be expressed in a series (which terminates under certain conditions) as

$$g_n = e^{-\xi} \cdot \left[1 - \frac{n}{\frac{1}{2}}\xi + \frac{n(n-1)}{\frac{1}{2}(\frac{1}{2}+1)} \cdot \frac{\xi^2}{2!} - \frac{n(n-1)(n-2)}{\frac{1}{2}(\frac{1}{2}+1)(\frac{1}{2}+2)} \cdot \frac{\xi^3}{3!} + \ldots\right] \quad (14)$$

where

$$n = \lambda' - \tfrac{1}{2} \quad (15)$$

Now it can be shown that, if n is zero or a positive integer, $g_n \to e^{-\xi} \cdot \xi^n$ as $\xi \to \infty$, thus satisfying the boundary condition $u_1 \to 0$ as $y \to \infty$. For negative values of n, $g_n \to \infty$ as $\xi \to \infty$, whereas for nonintegral positive values of n, $g \to \xi^{-\lambda'}$. Hence the only permissible functions are those corresponding to positive integral values of n. The solution given by Eq. (14) now may be written as

$$g_n = e^{-\eta^2} \cdot [H_{2n}(\eta)/H_{2n}(0)] \quad (16)$$

where $H_{2n}(\eta)$ are Hermite polynomials provided that $n = \lambda - (\sigma c_1/4\rho\nu) - \frac{1}{2}$ is a positive integer.

Since Eq. (4) is linear, a general solution can be written as

$$u_1 = \sum_{n=0}^{\infty} A_n \cdot \left(\frac{x}{x_0}\right)^{-[1/2 + \sigma c_1/4\rho\nu] - n} \cdot g_n(\xi) \quad (17)$$

which can be made to fit to some initial condition $u_{10}(y) = u_1(x_0, y)$ at the station $x = x_0$ where $C_0 = [4\nu x_0/U]^{1/2}$. Thus, u_{10} becomes a function of ξ, and Eq. (17) gives

$$u_{10}(\xi) = \sum_{n=0}^{\infty} A_n\, g_n(\xi) \quad (18)$$

It can be shown from Eq. (12) that the functions $g_n(\xi)$ are orthogonal in the interval 0 to ∞ with respect to the weighting function $\xi^{-1/2} \cdot e^{\xi}$. Thus the coefficients A_n may be written as

$$A_n = \frac{\int_0^{\infty} u_{10}(\xi) \cdot \xi^{-1/2} e^{\xi} g_n d\xi}{\int_0^{\infty} \xi^{-1/2} e^{\xi}\, g_n^2\, d\xi} \quad (19)$$

It is clear from (17) that the scale of the velocity defect inside the wake dies more quickly in the presence of a magnetic field than when the field is absent. Physically, this result is to be expected, as the effect of a transverse magnetic field is to make the velocity profile flatter and more uniform as in the case of Hartmann flow between two parallel plates. It also may be noted that the flux of momentum

$$\rho U \int_{-\infty}^{\infty} u dy$$

which is a measure of the resistance of the body is not independent of x but varies as $x^{-\sigma c_1/4\rho\nu}$ at large distance from the body, since $u_1 \sim x^{-[1/2 + \sigma c_1/4\rho\nu]} \cdot e^{-\eta^2}$ there. The invariance of the flux of momentum in the nonmagnetic case with respect to x is a consequence of the fact that the pressure is assumed constant inside the wake at a large distance from the body [although strictly speaking, inside the wake $p = P - (c_1 x/r^2) + 0(1/r^2)$, P being the freestream pressure], and this is not so in the present case due to the presence of the variable magnetic field, giving rise to a variable magnetic pressure. For a self-propelled body, $u_1 \sim x^{-[3/2 + (\sigma c_1/4\rho\nu)]} \cdot H_2(\eta)$, $H_2(\eta)$ being a Hermite polynomial.

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Local Solutions to the Two-Body Problem

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A nearly exact solution to the problem of two-bodies is given about a local point, in terms of the deviation in radial distance from the initial radius. When the deviation is small, as in the case of small eccentricity, the solution is competitive with that achieved by solving Kepler's equation. For higher eccentricities the solution holds with restraint on the time interval, without the need for specific attention to the crossover to parabolic and hyperbolic orbits. Two forms of the solution are manifested: one in terms of circular functions; the other in terms of hyperbolic functions.

Introduction

The problem of two-bodies begins in effect with the attempt to solve a nonlinear differential equation. Many ingenious approaches have evolved; but, unfortunately, the character of the equation leads inevitably to a singular solution at the parabolic orbit. Indeed, the failure of unembellished solutions to Kepler's equation to provide the means for precision orbit determination has attracted the efforts of the most brilliant minds in mathematics and astronomy. Expansions about Barker's solution in parabolic orbit are legion.

In the following paper is an attempt to solve the two-body problem for radial distance in terms of the time by expanding r about r_0 and solving for the resultant Δr. This approach converts the original nonlinear equation to a linear equation when terms of degrees two and higher are neglected, yielding an excellent solution for small eccentricity orbits in terms of the circular functions; and it also yields excellent solutions for higher eccentricity orbits in terms of both the circular and hyperbolic functions, when smaller increments in Δr are taken.

It is of supreme importance to have a sound appreciation of the implications of this simple solution. Not only is it valid through the range of eccentricities, but it can easily be improved by solution of the equations containing the second- and third-degree terms, in terms of the Jacobi elliptic func-

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